A NOTE ON COHOMOLOGY WITH LIMITED TORSION

BY

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ABSTRACT

In this note we prove that if a class x in a torsion free (ordinary) cohomology ring of a topological space satisfies $x^p = py$ (p—a prime) than y^p is divisible by p. The proof and the applications of this statement are related to the theory of secondary operations.

0. Introduction

0.1. General remarks: In this note, we prove the following:

THEOREM: *Let X be a topological space, and let p be a prime. Suppose $\beta H^*(X,Z_p) = 0$. If $x \in H^{2n}(X,Z_{p^2})$ satisfies $x^p = py$, then $y^p = pz$ for some $z \in H^{2np^2}(X,Z_p)$.*

COROLLARY: *Let X be a topological space, let p be a prime, and let $G = Z$ or Z_{p^r}. Suppose the reduction $H^*(X, G) \to H^*(X, Z_p)$ is onto. If $x \in H^{2n}(X,G)$ satisfies $x^p = py$, then $y^p = pz$ for some $z \in H^{2np^2}(X,G)$.*

The interesting fact about these propositions is that the statements are concerned only with the module and ring structures of the cohomology. (Note that the condition $\beta H^*(X,Z_p) = 0$ is equivalent to $H^*(X,Z_{p^2})$ being free Z_{p^2} module. On the other hand, the proof involves relations between primary and secondary operations.

This note generalizes results of Hubbuck (see [3] lemma 1.3) who used k-theory operations in his study. As a matter of fact, the present note was motivated by his paper.

0.2. Method of proof. A detailed proof is carried out in section 1 for the case p-odd. The case $p = 2$ is discussed in Section 2.

In the first step of the proof of the theorem, it is shown (Proposition 1.2) that the mod-p reduction of y is detected by a secondary operation ϕ defined on x.

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(This operation is studied also in [5].) The second step shows that the mod-p reduction of y is zero by showing that $\mathscr{P}^{np}\phi$ can be decomposed (mod the ideal generated by $\beta H^*(\ ,Z_p)$) as $\mathscr{P}^1\beta\phi_1$ where ϕ_1 is a secondary operation associated with the relation $\mathscr{P}^{np-1}\mathscr{P}^n = 0$ (p-odd). These two steps are sufficient for the proof of the theorem.

1. The case p-odd. Let $h: K(Z_{p^2},2n) \to K(Z_p,2np)$ be given by $h^*\iota_{2np} = (\bar{\iota}_{2n})^p$, $\bar{\iota}_{2n} \in H^{2n}(K(Z_{p^2},2n),Z_p)$. Let $r: E \to K(Z_{p^2},2n)$ be the fibration induced by h from: $\Omega K(Z_p, 2np) \approx K(Z_p,2np-1) \to \mathscr{L}K(Z_p, 2np) \to K(Z_p, 2np)$ ($\mathscr{L}K(Z_p,2np)$—the path space of $K(Z_p,2np)$). Let $j: K(Z_p,2np-1) \to E$ be the inclusion of the fiber:

$$\begin{array}{ccc} & K(Z_p,2np-1) & \\ \swarrow j & & \searrow \\ E & \longrightarrow & \mathscr{L}K(Z_p,2np) \\ \downarrow r & & \downarrow \\ K(Z_{p^2},2n) & \xrightarrow{h} & K(Z_p,2np) \end{array}$$

Note that E is a loop space and let μ denote its loop addition.

1.1. THEOREM (L. Smith, [4] proposition 5.5 III): *As an algebra* $H^*(E,Z_p) \approx (H^*(K(Z_{p^2},2n),Z_p) /\!/ \operatorname{im} h^*) \otimes \operatorname{im} j^*$.

1.2. PROPOSITION *There exists* $v \in H^{2np}(E,Z_p2)$ *satisfying* $pv = (r^*\iota_{2n})^p$, $\iota_{2n} \in H^{2n}(K(Z_{p^2}, 2n), Z_{p^2})$ *and* $j^*v_0 = \beta\iota_{2np-1}$ *where* $v_0 \in H^{2np}(E,Z_p)$ *is the reduction of* v.

PROOF. Consider the following ladder of fibrations:

$$\begin{array}{ccc} E & \xrightarrow{j_0} & K\ Z_{p^2},2np) \\ \downarrow r & & \downarrow r_0 \\ K(Z_{p^2},2n) & \xrightarrow{h_0} & K(Z_{p^2},2np) \\ \downarrow h & & \downarrow g_0 \\ K(Z_p,2np) & \xrightarrow{i_1} & K(Z_p,2np)\times K(Z_p,2np+1) \end{array}$$

where i_1 is the injection; $g_0 = (g_0' \times g_0'')^\circ \Delta$, g_0 induced by the reduction $Z_{p^2} \to Z_p$ of homotopy groups and $g_0''^*(\bar{\iota}_{2np+1}) = \beta_2\iota_{2np}$; $h_0^*\iota_{2np} = (\iota_{2n})^p$. One can check that r_0 induces the multiplication by p of homotopy groups. $v = j_0^* \iota_{2np} \in H^{2np}(E, Z_{p^2})$ is the desired class.

1.3 PROPOSITION: *There exists* $v_1 \in H^{2np}(E, Z_p)$ *with*

$$j^* v_1 = \beta \iota_{2np-1} \text{ and } \mu^* v_1 = v_1 \otimes 1 + 1 \otimes v_1 + \lambda \sum_{a=1}^{p-1} \frac{1}{p} \binom{p}{a} (r^* \bar{\iota}_{2n})^b \otimes (r^* \bar{\iota}_{2n})^{p-a}, \qquad 0 \neq \lambda \in Z_p.$$

PROOF. Consider the fibration: $K(Z_p, 2np+1) \xrightarrow{j_1} E' \xrightarrow{r_1} K(Z_{p^2}, 2np+1)$ induced by $h_1: K(Z_p 2, 2n+1) \to K(Z_p, 2np+2)$ $h_1^* \iota_{2np+2} = \beta \mathscr{P}^n \ \bar{\iota}_{2n+1}$. Since h_1^* is a monomorphism in dim $\leqq 2n+2$ $j_1^* = 0$ in dim $\leqq 2np+1$ and $1 \otimes \iota_{2np} \in H^* (\Omega E_1, Z_p) = h^*(K(Z_{p^2}, 2n) \times K(Z_p, 2np), Z_p)$ is not in $\operatorname{im} \sigma^*$. By [2] theorem 5.14, it follows that $1 \otimes \iota_{2np}$ is not a primitive. This implies that $a^p \neq 0$ for $0 \neq a \in H_{2n}(\Omega E_1, Z_p)$ and hence, there exists a class $u \in H^{2np}(\Omega E, Z_p)$ with $\bar{\mu}_0^* u = \sum_{a=1}^{p-1} \frac{1}{p} \binom{p}{a} (\Omega r_1^* \iota_{2n})^a \otimes (\Omega r_1 \iota_{2n})^{p-a}$ and $u = u' \otimes 1 + 1 \otimes \iota_{2np}$. Choosing a different representation of ΩE_1 as a cartesian product $K(Z_{p^2}, 2n) \times K(Z_p, 2np)$ one may assume that

$$\bar{\mu}_0^*(1 \otimes \iota_{2np}) = \sum_{a=1}^{p-1} \frac{1}{p} \binom{p}{a} (\Omega r_1^* \iota_{2n})^a \otimes (\Omega r_1^* \iota_{2n})^{p-a}.$$

Now, we have the following diagram:

$$\begin{array}{ccc}
E' & \xrightarrow{h_1'} & E_1 \\
\downarrow & \swarrow & \\
K(Z_{p^2}, 2n+1) & & \\
\downarrow h_2 & \searrow h_1 & \\
K(Z_p, 2np+1) & \longrightarrow & K(Z_p, 2np+2)
\end{array}$$

and $\Omega E' = E$; $h_2^* \iota_{2np+1} = \mathscr{P}^n \bar{\iota}_{2n+1}$. One can see that $v_1 = (\Omega h_1'^*)\ (1 \otimes \iota_{2np})$ is the desired class. Note that v_1^p is primitive and by Theorem 1.1, $v_0 - v_1$ is in the ideal generated by $\operatorname{im} r^*$.

1.4 LEMMA. *Let* $F \xrightarrow{i} E \xrightarrow{p} B$ *be a fibration.* $f: \Omega B \to F$ *the inclusion of the fiber of* i. *If* $t \in H^*(B, Z_p)$ *is in* $\ker p^*$, *then* $\sigma^* t \in H^*(\Omega B, Z_p)$ *is in* $\operatorname{im} f^*$.

PROOF. Consider the mapping $g: B \to K(Z_p, |t|)$, with $g^* \iota = t$ and where $|t|$ denotes the dimnesion of t. Since $g \circ p \approx *$ we have the following ladder:

$$\begin{array}{ccccccc}
\Omega B & \xrightarrow{f} & F & \xrightarrow{i} & E & \xrightarrow{p} & B \\
\downarrow \Omega G & & \downarrow g_2 & & \downarrow g_1 & & \downarrow g \\
K(Z_p,|t|-1) & \longrightarrow & K(Z_p,|t|-1) & \longrightarrow & \mathscr{L}K(Z_p,|t| & (\longrightarrow & K(Z_p,|t|)
\end{array}$$

and $f^*g_2^*\iota_{|t|-1} = \sigma^*t$.

1.5. LEMMA. *There exists* $\omega \in PH^*(E,Z_p)$ *satisfying* $j^*\omega = \mathscr{P}^{np-1}\iota_{2np-1}$ *and* $\sigma^*\omega = 1 \otimes \mathscr{P}^{np-1}\iota_{2np-2} \in H^*(\Omega E, Z_p) = H^*(K(Z_{p^2}, 2n-1) \times K(Z_p, 2np-2), Z_p)$.

PROOF. Consider the fibration:

$$\begin{array}{ccccc}
K(Z_p,2np) & \xrightarrow{j_2} & E' & \xrightarrow{r_2} & K(Z_{p^2},2n+1) \\
& \searrow & \swarrow & & \downarrow \\
& & \mathscr{L}K(Z_p,2np+1) & \longrightarrow & K(Z_p,2np+1)
\end{array}$$

$h_2^*\iota_{2np+1} = \mathscr{P}^n\iota_{2n+1}, \quad \Omega E' = E$.

Since $\mathscr{P}^{np-1}\mathscr{P}^n = 0$ $\mathscr{P}^{np-1}\iota_{2np+1} = 0$ and by 1.4 there exists $\omega' \in H^*(E',Z_p)$ satisfying $j_2^*\omega' = \mathscr{P}^{np-1}\iota_{2np}$. $\sigma^*\sigma^*\omega' = 1 \otimes \mathscr{P}^{np-1}\iota_{2np-2} + u' \otimes 1 \in H^*(\Omega\Omega E', Z_p)$ is a primitive and hence, $u' \in PH^*(K(Z_{p^2}, 2n-1), Z_p) \subset \operatorname{im} \sigma^*\sigma^*$. Altering ω' by element in $\operatorname{im} r_2^*$ we have $\sigma^*\sigma^*\omega' = 1 \otimes \mathscr{P}^{np-1}\iota_{2np-2}$ and $\omega = \sigma^*\omega'$ is the desired class.

1.6. PROPOSITION. *Let* $\xi: H^*(E,Z_p) \to H^*(E, Z_p)$ *be the p-th power operation* $\xi t = t^p$. *Then*

$$\mathscr{P}^1\beta\omega - v_1^p \in r^*(\operatorname{im}\xi) = \xi(\operatorname{im} r^*).$$

PROOF. $j^*(\mathscr{P}^1\beta\omega - v_1^p) = (\mathscr{P}^1\beta\mathscr{P}^{np-1} - \mathscr{P}^{np}\beta)\iota_{2n-p-1} = 0$. Hence, by Theorem 1.1, $\mathscr{P}^1\beta\omega - v_1^p$ is in the ideal generated by $\overline{\operatorname{im} r^*}$. But $\mathscr{P}^1\beta\omega - v_1^p$ being a primitive and since $H^*(K(Z_{p^2}, 2n), Z_p)$ and im r^* are primitively generated, this implies that $\mathscr{P}^1\beta\omega - v_1{}^p \in r^*(PH^*(K(Z_{p^2}, 2n), Z_p))$. Finally $\sigma^*(\mathscr{P}^p\beta\omega - v_1^p) = 1 \otimes \mathscr{P}^1\beta P^{np-1}\iota_{2np-2} = 0$, hence $\mathscr{P}^1\beta\omega - v_1^p \in r^*[(PH^{2np}K(Z_{p^2}, 2n) \cap \ker \sigma^*] \subset r^*(\operatorname{im}\xi)$.

1.7 PROPOSITION. $v_0^p - \mathscr{P}^1\beta\omega$ *is in the ideal generated by*

$$r^*[\beta\bar{H}^*(K(Z_{p^2}, 2n), Z_p)].$$

PROOF. $v_1^p - v_0^p = \xi(v_1 - v_0)$ and $\mathscr{P}^1\,\beta\omega - v_1^p$ are in the ideal generated b $\xi(\operatorname{im} r^*)$. But since $\xi\mathscr{P}^j\bar{\iota}_{2n} = \mathscr{P}^{pj}\xi\bar{\iota}_{2n} \in \ker r^*$, $\xi(\operatorname{im} r^*) = \xi r^* A$ where $A \subset H^*(K(Z_{p^2}, 2n), Z_p)$ is the algebra generated by $\beta\bar{H}^*(K(Z_p 2, 2n), Z_p)$.

1.8. THEOREM. *Let X be a topological space, p—an odd prime, and $x \in H^{2n}(X, Z_{p^2})$. If $\beta H^*(X, Z_p) = 0$ and $x^p = py$, then $y^p = pz$ for some $z \in H^{2np^2}(X, Z_{p^2})$.*

PROOF. Let $f: X \to K(Z_{p^2}, 2n)$ be given by $f^*\iota_{2n} = x$. f can be lifted to $\tilde{f}: X \to E$ and by 1.2 $x^p = p\tilde{f}^*v$. Hence, $y - \tilde{f}^*v$ has order p. But as the Bockstein exact sequence for X yields a short exact sequence,

$$0 \to H^*(X, Z_p) \to H^*(X, Z_{p^2}) \to H^*(X, Z_p) \to 0,$$

$H^*(X, Z_p 2) \xrightarrow{\times p} H^*(X, Z_{p^2}) \to H^*(X, Z_p) \to 0$ is exact and $y - \tilde{f}v = pz_1$ for some $z_1 \in H^{2np}(X, Z_{p^2})$. Moreover, by 1.7, $(\tilde{f}^*v_0)^p$ is in the ideal generated $\beta \bar{H}^*(X, Zp) = 0$, hence $(\tilde{f}^*v_0)^p = 0$ and again $(\tilde{f}^*v)^p = pz_2$, $z_2 \in H^{2np^2}(X, Z_{p^2})$. It follows that $(pz_1)^p = (y - \tilde{f}^*v)^p = y^p - pz_2 + pz_3$ and 1.8 follows.

1.9. COROLLARY *Let X be a topological space, p—an odd prime, and let G be either Z or Z_{p^r}. Suppose $H^*(X, G) \to H^*(X, Z_p)$ is onto. If $x \in H^{2n}(X, G)$ satisfies $x^p = py$ then $y^p = pz$ for some $z \in H^{2np^r}(X, G)$.*

In order to prove 1.9, we first prove the following:

1.10. LEMMA. *If $H^*(X, Z_{p^r}) \to H^*(X, Z_p)$ is onto, then*

(a) $H^*(X, Z_{p^r}) \xrightarrow{\times p} H^*(X, Z_{p^r}) \xrightarrow{\alpha_r} H^*(X, Z_p) \to 0$ *is exact.*

(b) $h^*(X, Z_{p^v}) \xrightarrow{\alpha_{r,t}} H^*(X, Z_{p^t})$ *is onto for all* $1 \leqq t \leqq r$.

PROOF. We first note that $\alpha_t: H^*(X, Z_p t) \to H^*(X, Z_p)$ is onto for all $1 \leqq t \leqq r$. Hence, the Bockstein long exact sequence decomposes into short exact sequences: $0 \to H^*(X, Z_{p^{t-1}}) \to H^*(X, Z_{p^t}) \to H^*(X, Z_p) \to 0$. If $(b)_t$ $H^*(X, Z_{p^t}) \to H^*(X, Z_{p^{t-1}})$ is onto, then it follows that:

$(a)_t$ $$H^*(X, Z_{p^t}) \xrightarrow{\times p} H^*(X, Z_{p^t}) \xrightarrow{\alpha_t} H^*(X, Z_p) \to 0$$

is exact.

Suppose $(a)_t$ holds.

Consider now $\alpha_{t+1,t}$: $H^*(X, Z_{p^{t+1}}) \to H^*(X, Z_{p^t})$. If $x \in H^*(X, Z_{p^t})$ then, for some $h_1 \in H^*(X, Z_{p^{t+1}})$ $x - \alpha_{t+1,t}y_1 \in \ker \alpha_t$ and hence, by $(a)_t$, $x = \alpha_{t+1,t}y_1 + px_1$ for some $x_1 \in H^*(X, Z_{p^t})$. Similarly,

$$x_1 = \alpha_{t+1,t}y_2 + px_2$$

$$x_{t-1} = \alpha_{t+1,t}y_t + px_t$$

$$x_i \in H^*(X, Z_{p^t})$$

Therefore,

$$x = \alpha_{t+1,t}(y_1 + py_2 + \cdots + p^{t-1}y_t) + p^t x_t$$

and as $p^t x_t = 0$, it follows that

$$(b)_{t+1}\alpha_{t+1,t}\colon H^*(X, Z_{p^{t+1}}) \to H^*(X, Z_{p^t})$$

is onto and the lemma follows from the inductive argument $(a)_t \Rightarrow (b)_{t+1} \Rightarrow (a)_{t+1}$ ($(b)_1$ obviously holds.)

Proof of Corollary. If $x^p = py$, then the image of y^p in $H^*(X, Z_p)$ is zero. This follows from 1.8 by reducing x and y to $H^*(X, Z_{p^2})$. But then, if $G = Z_{p^r}$, by 1.10 (a) $y^p = pz$ for some $z \in H^{2np^2}(X, Z_{p^r})$ The case $G = Z$ follows similarly from the exact sequence $0 \to H^*(X,Z) \overset{\times p}{\to} H^*(X,Z) \to H^*(X,Z_p) \to 0$

2. The case $p = 2$ and general remarks. The only proposition in section 1 which fails to hold after replacing $\mathcal{P}^k$ by Sq^{2k} is 1.5 as $Sq^{4n-2}Sq^{2n} \neq 0$.

Instead we have $Sq^{4n-2}Sq^{2n} + Sq^{4n-1}Sq^{2n-1} = 0$. To overcome this difficulty, we replace the "universal example" E by $\tilde{E}$ obtained as the fiber of $\tilde{h}\colon K(Z_4,2n) \to K(Z_2,4n) \times K(Z_2,4n-1)$ satisfying $\tilde{h}^*\iota_{4n} = \iota_{2n}, \tilde{h}^*\iota_{4n-1} = Sq^{2n-1}\iota_{2n}$. We then have a class $\tilde{\omega} \in H^*(\tilde{E}, Z_2)$ with $\tilde{j}^*\tilde{\omega} = Sq^{4n-2}\iota_{4n-1} \otimes 1$ where $\tilde{j}\colon K(Z_2, 4n-1) \times K(Z_2, 4n-2) \to \tilde{E}$ is the inclusion of the fiber. The mapping $f\colon X \to K(Z_4, 2n)$ realizing the class x can still be lifted to $\tilde{f}\colon X \to \tilde{E}$ as $Sq^{4n-1}x = Sq^1 Sq^{4n-2}x = 0$ and the rest of the arguments follow through.

We would like to remark that, in general, if $x^p = py$ it might happen that y is divisible by p and then 1.8 is essentially void. This, however, cannot happen if X is an H-space and $x_1^p = 0$ mod-p yields a (non zero mod-p) class x_2 with $x_2^p = 0$ mod-p and the procedure yields an ∞ tower of elements $x_n, x_n^p = 0$, x_n being a (mod-p) 1-implication of x_n (in the sense of W. Browder, see [1] page 357).

References

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